The anisotropic cross-section for the spherical Gaussian medium

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Abstract

We investigate new derivations of the anisotropic cross-sections (the distribution of visible normals) for a variety of surface and volumetric media.

1 Intro

In this brief we consider the problem of deriving anisotropic cross sections for a variety of anisotropic media, inspired by the proposal to unify rough surface scattering and volumetric scattering in the form proposed by [Heitz et al. 2016a]—the holy grail being the derivation of the cross section for the spherical Gaussian normal distribution. We remain suspicious that a simple compact form for this cross section is possible, but have failed to find it, leaving it as a challenge to the reader.

1.1 Scope

Given a normal distribution function on the sphere

$$D(\omega) \tag{1}$$

we seek to derive the distribution of normals visible from incoming direction ω_i (equivalently, the anisotropic cross-section for a volumetric media):

$$\sigma = \int_{4\pi} D(\omega)\langle\omega_i, \omega\rangle d\omega \tag{2}$$

where $\langle\omega_i, \omega\rangle$ is the clamped cosine,

$$\langle\omega_i, \omega\rangle = \max(0, \omega \cdot \omega_i). \tag{3}$$

1.2 Symmetry

We consider only distributions with azimuthal symmetry and with the notation $u = \cos\theta$, specifying, for example, the isotropic distribution of normals as

$$D(u) = \frac{1}{2} \tag{4}$$

with normalization

$$\int_{-1}^{1} D(u)du = 1. \tag{5}$$

2 Cross-section derivations

2.1 Dirac normal distributions

The simple case of a flat surface, with a Dirac delta normal distribution,

$$D(u) = \delta(u-1) \tag{6}$$

produces, through straightforward derivations, the simple cross-section

$$\sigma = \frac{1}{2}\left(u + |u|\right), \tag{7}$$

which is an important limiting case for all of the parametric normal distributions we consider here in the case that roughness approaches zero.

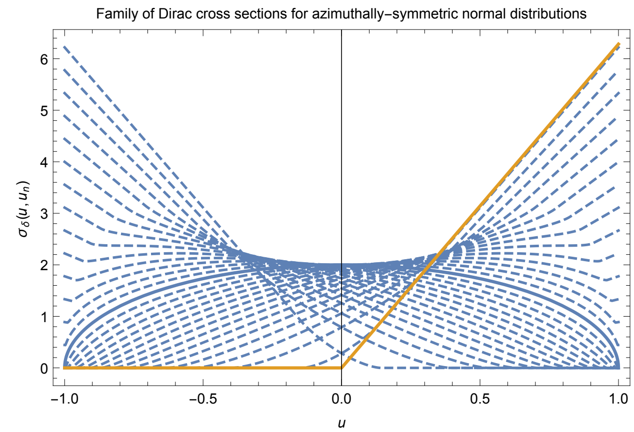

Figure 1: *The Dirac-delta cross sections $\sigma_\delta(u, u_n)$ for a variety of u_n form a family of normalized functions blending between $\pi(u + |u|)$ and $2\sqrt{1-u^2}$.*

The general case of a dirac ring of normals with azimuthal symmetry and all normals inclined by a common cosine, u_n,

$$D(u) = \delta(u - u_n), \tag{8}$$

leads to a piecewise simple expression for the cross-section,

$$\sigma_\delta(u, u_n) = \begin{cases} 2\pi u u_n & \left(u_n < 0 \land u < -\sqrt{1-u_n^2}\right) \lor \left(u_n > 0 \land u > \sqrt{1-u_n^2}\right) \\ 2\left(\sqrt{-u_n^2 - u^2 + 1} + u u_n \cos^{-1}\left(-\frac{u u_n}{\sqrt{1-u^2}\sqrt{1-u_n^2}}\right)\right) & u > -\sqrt{1-u_n^2} \land u < \sqrt{1-u_n^2} \end{cases},$$

which we will later find convenient to divide into three component functions,

$$\sigma_\delta = \sigma_{\delta 1} + \sigma_{\delta 2} + \sigma_{\delta 3} \tag{9}$$

$$\sigma_{\delta 1} = \begin{cases} 2\pi u u_n & u u_n > 0 \\ 0 & else \end{cases} \tag{10}$$

$$\sigma_{\delta 2} = \begin{cases} 2\sqrt{-u_n^2 - u^2 + 1} & |u| < \sqrt{1-u_n^2} \\ 0 & else \end{cases} \tag{11}$$

$$\sigma_{\delta 3} = \begin{cases} 2 u u_n \cos^{-1}\left(-\frac{u u_n}{\sqrt{1-u^2}\sqrt{1-u_n^2}}\right) - 2\pi u u_n \theta(u u_n) & |u| < \sqrt{1-u_n^2} \\ 0 & else \end{cases} \tag{12}$$

where $\theta(x)$ is the Heaviside theta function. This produces an interesting family of symmetric,

$$\sigma_\delta(a, b) = \sigma_\delta(b, a), \tag{13}$$

normalized,

$$\int_{-1}^{1} \sigma_\delta(u, u_n)du = \pi \tag{14}$$

functions.

The cross section of any symmetric normal distribution $D(u)$ can then be found as an integral of these dirac cross-sections,

$$\sigma = \int_{-1}^{1} \sigma_\delta(u, u_i)D(u)du. \tag{15}$$

2.1.1 Polynomial distributions

Odd powers of u have a simple cross section,

$$\int_{-1}^{1} \sigma_\delta(u,u_i) u^p du = \frac{2\pi}{p+2} u_i, \quad \text{(p odd)} \tag{16}$$

and, thus, the cross section of any distribution is always an even function plus a linear function (of u_i).

For even powers of u, some of the dirac component integrals have simple expressions

$$\int_{-1}^{1} \sigma_{\delta 1}(u,u_i) u^p du = \frac{\pi}{p/2+1} |u_i|, \quad \text{(p even)}. \tag{17}$$

$$\int_{-\sqrt{1-u_i^2}}^{\sqrt{1-u_i^2}} (-2\pi u u_i \theta(u u_i)) u^p du = \frac{(-1)^{p/2} \pi (u_i-1)^{p/2+1}}{p/2+1} |u_i|, \quad \text{(p even)}. \tag{18}$$

Table 1 details additional exploration of the dirac delta component integrals for a variety of basic distributions.

2.2 Beckmann

The Beckmann normal distribution with roughness α is given by

$$D_B(u) = \begin{cases} \frac{e^{-\frac{1}{\alpha^2}\left(\frac{1}{u^2}-1\right)}}{\pi\alpha^2 u^4} & u > 0 \\ 0 & \text{True} \end{cases} \tag{19}$$

The cross section has been presented previously [Walter et al. 2007; Heitz 2014] (it is related to Smith's Lambda function by a cosine factor [Heitz et al. 2016b]), derived using the distribution of slopes,

$$\sigma = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\left(u_i - p\sqrt{1-u_i^2}\right)\theta\left(u_i - p\sqrt{1-u_i^2}\right)}{\sqrt{p^2+q^2+1}} (D(u)+D(-u)) u^3 \, dp\, dq \tag{20}$$

with the change of variables

$$u = \frac{1}{\sqrt{p^2+q^2+1}}, \tag{21}$$

yielding

$$\sigma = \frac{1}{2}\left(u_i \left[1 + \operatorname{erf}\left(\frac{u_i}{\alpha\sqrt{1-u_i^2}}\right)\right] + \frac{\alpha\sqrt{1-u_i^2}\, e^{\frac{u_i^2}{\alpha^2(u_i^2-1)}}}{\sqrt{\pi}} \right). \tag{22}$$

The even function

$$\sigma_2 = \int_{-1}^{1} \sigma_{\delta 2}(u,u_i) D_B(u) du = \frac{1}{2} \frac{\alpha\sqrt{1-u_i^2}\, e^{\frac{u_i^2}{\alpha^2(u_i^2-1)}}}{\sqrt{\pi}} \tag{23}$$

is produced by the second Dirac delta component.

2.3 GGX

The GGX distribution of roughness α is given by

$$D_{GGX}(u) = \begin{cases} \frac{\alpha^2}{\pi u^4 \left(\alpha^2 + \frac{1}{u^2} - 1\right)^2} & u > 0 \\ 0 & \text{True} \end{cases}. \tag{24}$$

The cross section has also been previously derived using the distribution of slopes,

$$\sigma = \frac{1}{2}\left(\sqrt{\alpha^2 - \alpha^2 u_i^2 + u_i^2} + u_i\right). \tag{25}$$

2.4 Kagiwada-Kalaba / Ellipsoidal

In this section we consider a distribution of normals inspired by the Kagiwada-Kalaba phase function [1967] (also known as ellipsoidal [Sobolev 1975]),

$$D_{KK}(u) = \frac{b}{(1-bu)\log\left(\frac{1+b}{1-b}\right)} \tag{26}$$

with shape parameter $0 < b < 1$, which produces the isotropic distribution as $b \to 0$ and a flat surface as $b \to 1$.

We were able to derive the cross-section by integrating a uniform disk of parallel rays inclined along u_i and intersecting the sphere,

$$\begin{aligned} \sigma &= \pi \int_0^1 \int_0^1 D_{\mathrm{KK}}\left(\cos(2\pi\xi_2)\sqrt{\xi_1 - \xi_1 u_i^2} + \sqrt{1-\xi_1}\, u_i\right) d\xi_2 d\xi_1 \\ &= \frac{2\pi\left(\sqrt{b^2\left(u_i^2-1\right)+1} + u_i \log\left(\sqrt{b^2\left(u_i^2-1\right)+1} - u_i\right) + b u_i - u_i \log\left(\left(-b-1\right)\left(u_i-1\right)\right) - 1\right)}{b \log\left(\frac{2}{b+1} - 1\right)}. \end{aligned} \tag{27}$$

2.5 Spherical Gaussian

The canonical extension of the Beckmann normal distribution to the entire sphere is the spherical Gaussian (von-Mises Fischer) distribution,

$$D_{SG}(u) = \frac{\operatorname{csch}\left(\frac{2}{\alpha^2}\right) e^{\frac{2u}{\alpha^2}}}{2\pi\alpha^2}, \tag{28}$$

where we have chosen shape parameter α such that $D_{SG}(u)$ is asymptotic to $D_B(u)$ for small α. For large roughnesses the two distributions differ with D_{SG} approaching the isotropic distribution on the entire sphere as roughness $\alpha \to \infty$.

We were unable to derive a simple closed form for the spherical Guassian cross section. We encourage the reader to find progress towards this goal where we have failed.

A uniform disk of parallel rays yields,

$$\boxed{\sigma = \frac{\operatorname{csch}\left(\frac{2}{\alpha^2}\right)}{2\alpha^2} \int_0^1 e^{\frac{2\sqrt{1-\xi}u_i}{\alpha^2}} \, {}_0\tilde{F}_1\left(;1;\frac{\xi - \xi u_i^2}{\alpha^4}\right) d\xi} \tag{29}$$

where ${}_0\tilde{F}_1$ is the regularized confluent hypergeometric function.

End and midpoints for σ are

$$\sigma(-1) = \frac{1}{4}\left(\alpha^2 - e^{\frac{2}{\alpha^2}}\left(\alpha^2-2\right)\right)\operatorname{csch}\left(\frac{2}{\alpha^2}\right) \tag{30}$$

$$\sigma(0) = \frac{1}{2}\operatorname{csch}\left(\frac{2}{\alpha^2}\right) I_1\left(\frac{2}{\alpha^2}\right) \tag{31}$$

$$\sigma(1) = \frac{1}{4}\left(\alpha^2 - e^{-\frac{2}{\alpha^2}}\left(\alpha^2+2\right)\right)\operatorname{csch}\left(\frac{2}{\alpha^2}\right). \tag{32}$$

The normal distribution can be decomposed,

$$D_{SG}(u) = \frac{\operatorname{csch}\left(\frac{2}{\alpha^2}\right)\cosh\left(\frac{2u}{\alpha^2}\right)}{2\pi\alpha^2} + \frac{\operatorname{csch}\left(\frac{2}{\alpha^2}\right)\sinh\left(\frac{2u}{\alpha^2}\right)}{2\pi\alpha^2} \tag{33}$$

The cross section is the sum of an even function and a linear function

$$\boxed{\sigma = \sigma_{odd} + \sigma_{even}} \tag{34}$$

with

$$\boxed{\sigma_{odd} = \frac{1}{4}\left(2\coth\left(\frac{2}{\alpha^2}\right) - \alpha^2\right) u_i} \tag{35}$$

and

$$\boxed{\sigma_{even} = \operatorname{csch}\left(\frac{2}{\alpha^2}\right) \sum_{j=0}^{\infty} \frac{(-1)^{j+1} \alpha^{-2j} I_{j+1}\left(\frac{2}{\alpha^2}\right)}{2(2j-1)j!} u_i^{2j}} \tag{36}$$

$$\sigma_2 = \frac{\operatorname{csch}\left(\frac{2}{\alpha^2}\right)\left(1-u_i^2\right) {}_0\tilde{F}_1\left(;2;\frac{1-u_i^2}{\alpha^4}\right)}{2\alpha^2} \tag{37}$$

A spherical harmonics expansion leads to [Aubry 2016] the following expression for the cross section

$$\boxed{\sigma = \sum_{l=0}^{\infty} \frac{\pi^{3/2}(2l+1)c_l P_l\left(u_i\right)}{4\Gamma\left(\frac{3}{2}-\frac{l}{2}\right)\Gamma\left(\frac{l}{2}+2\right)}} \tag{38}$$

$$c_l = \int_{-1}^{1} D_{SG}(u) P_l(u) du \tag{39}$$

where P_l are the Legendre polynomials.

2.5.1 Approximate Forms

We found several efficient approximate forms for computing the spherical Gaussian cross section.

The Beckmann cross section (Equation 20) is a reasonable approximation for the spherical Guassian cross section for very low roughness, accurate to within 0.001 for roughness $m < 0.044$.

An easy improvement on this approximation, accurate to within 0.001 for roughness $m < 0.25$ is found by correcting the above approximation to satisfy the known exact odd component of the solution,

$$\sigma \approx \frac{1}{4}\left(\tanh\left(\frac{1}{m^2}\right)\left(u - m^2\left|u\right|\right) + \left(-m^2-2\right)u + u\coth\left(\frac{1}{m^2}\right)\right) + \frac{1}{2}\left(u\operatorname{erf}\left(\frac{u}{m\sqrt{1-u^2}}\right) + \frac{m\sqrt{1-u^2}e^{\frac{u^2}{m^2\left(u^2-1\right)}}}{\sqrt{\pi}} + u\right). \tag{40}$$

A further improvement for low roughness, accurate to within 0.0001 for roughness $m < 0.3$ is,

$$\sigma \approx \frac{1}{4}\left(2\coth\left(\frac{2}{m^2}\right) - m^2\right)\left(u_i\operatorname{erf}(A-B)+1\right) - \frac{\left(u_i^2-1\right)\operatorname{csch}\left(\frac{2}{m^2}\right) {}_0\tilde{F}_1\left(;2;\frac{1-u_i^2}{m^4}\right)}{2m^2} \tag{41}$$

$$A = \frac{\sqrt{\pi}u_i^3\operatorname{csch}\left(\frac{2}{m^2}\right)\left(m^{12}\left(2\,{}_0\tilde{F}_1\left(;2;\frac{1}{m^4}\right) - {}_0\tilde{F}_1\left(;3;\frac{1}{m^4}\right)\right)\left(m^2-2\coth\left(\frac{2}{m^2}\right)\right)^2 - \pi\left(m^4\,{}_0F_1\left(;2;\frac{1}{m^4}\right) + {}_0F_1\left(;3;\frac{1}{m^4}\right)\right)^3\operatorname{csch}^2\left(\frac{2}{m^2}\right)\right)}{3m^{18}\left(m^2-2\coth\left(\frac{2}{m^2}\right)\right)^3} \tag{42}$$

$$B = \frac{\sqrt{\pi}\left(m^4\,{}_0F_1\left(;2;\frac{1}{m^4}\right) + {}_0F_1\left(;3;\frac{1}{m^4}\right)\right)u_i\operatorname{csch}\left(\frac{2}{m^2}\right)}{m^8-2m^6\coth\left(\frac{2}{m^2}\right)} \tag{43}$$

For high roughness, a low order spherical harmonic expansion is accurate to within 0.001 for $m > 1$,

$$\sigma \approx -\frac{1}{4}u_i\left(m^2-2\coth\left(\frac{2}{m^2}\right)\right) + \frac{5}{128}\left(3u_i^2-1\right)\left(3m^4-6m^2\coth\left(\frac{2}{m^2}\right)+4\right) + \frac{1}{4}. \tag{44}$$

A useful approximation in between the above high and low roughness approximations is the spherical harmonics expansion (Equation 38) truncated to 12 terms.

3 Conclusion

The extension of normal distributions onto the entire sphere for defining new surface-like and semi-porous reflectance behaviours from semi-infinite half spaces as proposed by [Heitz et al. 2016a] requires two steps, derivation of the scattering cross section and sampling of the distribution of visible normals. We have undertaken the first step for several plausible extensions of height field normal distribution functions, the ellipsoidal and spherical Guassian distributions, each of which have the desired property of blending continuously between a flat mirror surface at one extreme and an isotropically-scattering half-space at the other. We were only able to derive a simple closed form cross section for the ellipsoidal distribution. We explored the derivation of the desirable spherical Gaussian cross section in detail providing a variety of expansions and approximations and leaving the task of finding a simpler closed form as a challenge to the reader.

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Distribution $D(u)$	σ_1	σ_2	σ_3	$\sigma = \sigma_1 + \sigma_2 + \sigma_3$
1	$\pi\lvert u\rvert$	$\pi(1-u^2)$	$\pi(u^2-\lvert u\rvert)$	π
u	$\frac{2\pi}{3}u$	0	0	$\frac{2\pi}{3}u$
u^2	$\frac{\pi}{2}\lvert u\rvert$	$\frac{1}{4}\pi(-1+u^2)^2$	$-\frac{1}{4}\pi\left(2\lvert u\rvert+u^4-3u^2\right)$	$\frac{1}{4}\pi\left(u^2+1\right)$
u^4	$\frac{\pi}{3}\lvert u\rvert$	$-\frac{1}{8}\pi(-1+u^2)^3$		$-\frac{1}{24}\pi\left(u^4-6u^2-3\right)$
u^p (p even)	$\frac{2\pi}{p+2}\lvert u\rvert$	$(-1)^{p/2+1}\pi\binom{p}{p/2}\frac{2-p}{p/2+1}(-1+u^2)^{p/2+1}$		
u^p (p odd)	$\frac{2\pi}{p+2}u$	0	0	$\frac{2\pi}{p+2}u$
$\frac{1}{1+u}$		$2\pi(1-\lvert u\rvert)$		
$\frac{1}{1-u}$		$2\pi(1-\lvert u\rvert)$		
$\frac{1}{(1-u)^2}$		$2\pi(-1+\frac{1}{\lvert u\rvert})$		
$\frac{1}{1+u^2}$	$\pi\lvert u\rvert\log 2$	$2\pi(-1+\sqrt{2-u^2})$		

Table 1: *Component integrals for various distributions*