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Radiative transfer in half spaces of arbitrary dimension

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We generalize the classic albedo and Milne problems for half spaces with isotropic scattering to Euclidean domains of general dimension $d \geq 1$ by defining measure $G(\mu)d\mu$ for angle cosine $-1 \leq \mu \leq 1$,

$$G(\mu) = \frac{2(1-\mu^2)^{\frac{d-3}{2}}\Gamma(\frac{d}{2})}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)}, \quad d > 1. \tag{1}$$

The plane-parallel linear transport equation in a half space in $\mathbb{R}^d$ becomes

$$\left(\mu\frac{\partial}{\partial x} + 1\right) I(x,\mu) = \frac{c}{2}\int_{-1}^{1} I(x,\mu')G(\mu')d\mu'. \tag{2}$$

This forms a pseudo problem with characteristic $\Psi(\mu) = (c/2)G(\mu)$. The classic 3D and ocassionally-practical 2D "Flatland" domains are included in a new general family of solutions with $G(\mu) = 1$ for 3D, and $G(\mu) = 2/(\pi\sqrt{1-\mu^2})$ for Flatland. We find closed-form solutions of the general transport equation in terms of hypergeometric functions that paint the classic 3D albedo problem in a new light and unify three common Wiener-Hopf kernels [2].

The corresponding Wiener-Hopf integral equation for the scalar collision-rate density $C(x)$ at optical depth x is

$$C(x) = C_0(x) + c\int_0^{\infty} K(x-x')C(x')dx', \tag{3}$$

with density of initial collisions $C_0(x)$, single-scattering albedo c, and displacement kernels

$$K(x) = \frac{1}{2}\int_0^1 e^{-|x|/\mu}\frac{1}{\mu}G(\mu)d\mu = \frac{1}{2\pi}\Gamma\left(\frac{d}{2}\right) G^{3,0}_{1,3}\left(\frac{x^2}{4}\,\middle|\, \begin{matrix}\frac{d-1}{2}\\ 0,0,\frac{1}{2}\end{matrix}\right) \tag{4}$$

given in the general case in terms of Meijer G functions. The Fourier transforms of the kernels are expressible in the general case using hypergeometric functions,

$$\tilde{K}(t) \equiv \int_{-\infty}^{\infty} K(x)e^{ixt}dx = {}_2F_1\left(\frac{1}{2},1;\frac{d}{2};-t^2\right), \tag{5}$$

that yield a dispersion equation

$$\Lambda(z) = 1 - \frac{c\,z}{2}\int_{-1}^{1}\frac{G(\mu)}{z-\mu}\,d\mu = 1 - c\;{}_2F_1\left(\frac{1}{2},1;\frac{d}{2};\frac{1}{z^2}\right), \tag{6}$$

which we find admits a finite eigenvalue $\nu_0 > 1$ if and only if

$$(d-3)/(d-2) < c < 1. \tag{7}$$

A related principal value integral is also generally expressible as a hypergeometric function,

$$\lambda(\nu) = 1 - \frac{c\nu}{2}\mathcal{P}\int_{-1}^{1} \frac{G(\mu)}{\nu - \mu}\, d\mu = 1 - c + c\,{}_2F_1\left(1, 1 - \frac{d}{2}; \frac{1}{2}; \nu^2\right). \tag{8}$$

The H functions for the Wiener-Hopf equation $H(z)H(-z) = 1/\Lambda(z)$ are given uniquely by [1]

$$H(z) = \exp\left(\frac{z}{\pi}\int_0^\infty \frac{1}{1+z^2t^2}\log\left[\frac{1}{1-c\,\tilde{K}(t)}\right]dt\right), \quad \mathrm{Re}\, z > 0 \tag{9}$$

with limits $H(0) = 1$ and $H(\infty) = (1-c)^{-1/2}$. Alternatively, defining

$$\tan\theta(t) = \frac{\pi t \Psi(t)}{\lambda(t)} = \frac{c}{2}\frac{\pi t G(t)}{\lambda(t)}, \tag{10}$$

choosing $\theta(0) = 0$, and ensuring $0 \leq \theta(t) \leq \pi$, we can also express H as

$$H(z) = \frac{1+z}{\nu_0 + z}\frac{1}{\sqrt{1-c}}\exp\left(-\frac{1}{\pi}\int_0^1 \frac{\theta(t)}{t+z}dt\right), \quad c > \frac{d-3}{d-2}, \tag{11}$$

or, for $c \leq (d-3)/(d-2)$, where no ν_0 exists,

$$H(z) = \frac{1}{\sqrt{1-c}}\exp\left(-\frac{1}{\pi}\int_0^1 \frac{\theta(t)}{t+z}dt\right), \quad z \notin [-1, 0]. \tag{12}$$

Additionally, for $(d-3)/(d-2) < c < 1$, the positive discrete eigenvalue $\nu_0 > 1$ is given by

$$\nu_0 = \pm\frac{1}{\sqrt{1-c}}\exp\left(-\frac{1}{\pi}\int_0^1 \frac{\theta(t)}{t}dt\right). \tag{13}$$

With H-function moments α_n defined by

$$\alpha_n = \int_0^1 \mu^n H(\mu) G(\mu)\, d\mu \tag{14}$$

we find that these moments satisfy the recurrence equations [4]

$$\alpha_{2n}\sqrt{1-c} = g_{2n} + \frac{c}{4}\sum_{k=1}^{2n-1}(-1)^k \alpha_{2n-k}\alpha_k, \quad n = 0, 1, 2, \ldots \tag{15}$$

where the characteristic moments g_{2n} are given by

$$g_{2n} = \int_0^1 \mu^{2n} G(\mu) d\mu = \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{d}{2}+n\right)}. \tag{16}$$

Low order moments for $0 < c < 1$ simplify to

$$\alpha_0 = \frac{2}{c}\left(1-\sqrt{1-c}\right), \quad \alpha_2 = \frac{1}{\sqrt{1-c}}\left(\frac{1}{d} - \frac{c}{4}\alpha_1^2\right). \tag{17}$$

Also, for $c = 1$, $\alpha_0 = 2$, and $\alpha_1 = 2/\sqrt{d}$. The equation for α_0 also yields a useful form of the integral equation for H,

$$\frac{1}{H(\mu)} = (1-c)^{1/2} + \frac{c}{2}\int_0^1 \frac{\mu' H(\mu')}{\mu+\mu'} G(\mu') d\mu'. \tag{18}$$

The law of diffuse reflection generalizes to [1]

$$I(0,\mu;-\mu_\ell) = \frac{c}{2}\frac{H(\mu)H(\mu_\ell)}{\mu+\mu_\ell}, \tag{19}$$

with diffuse albedo $R(\mu_\ell)$ under mono-directional illumination along cosine μ_ℓ given by

$$R(\mu_\ell) = \int_0^1 \mu I(0,\mu;-\mu_\ell) G(\mu) d\mu = 1 - \sqrt{1-c}H(\mu_\ell) \tag{20}$$

and under uniform diffuse illumination by

$$R = 1 - \alpha_1\sqrt{1-c}\frac{\sqrt{\pi}\Gamma((d+1)/2)}{\Gamma(d/2)}. \tag{21}$$

We also generalize Case's method [3] by separating variables with the ansatz

$$I(x,\mu) = \phi(\nu,\mu)\exp(-x/\nu), \tag{22}$$

and then imposing the normalization condition for the eigenmodes to be

$$\int_{-1}^1 \phi(\nu,\mu) G(\mu)\, d\mu = 1\,, \quad \nu \in \sigma, \tag{23}$$

where $\sigma = \{\nu \in [-1,1] \cup \pm\nu_0\}$ is the eigenvalue spectrum. The discrete eigenmodes, when they occur, satisfy

$$\phi(\pm\nu_0,\mu) = \frac{c\nu_0}{2(\nu_0 \mp \mu)}. \tag{24}$$

The eigenmodes for the continuum are

$$\phi(\nu,\mu) = \frac{c\nu}{2}\mathcal{P}\frac{1}{\nu-\mu} + \frac{\lambda(\nu)}{G(\nu)}\,\delta(\nu-\mu), \quad \nu \in [-1,1], \tag{25}$$

and the normalization integrals become

$$N(\nu_0) = \int_{-1}^1 \phi^2(\nu_0,\mu)\,\mu\, G(\mu)\, d\mu = \left.\frac{c\,\nu_0^2}{2}\frac{d\Lambda(z)}{dz}\right|_{z=\nu_0} = \frac{c^2\, {}_2F_1\left(\frac{3}{2}, 2; \frac{d}{2}+1; \frac{1}{\nu_0^2}\right)}{d\,\nu_0} \tag{26}$$

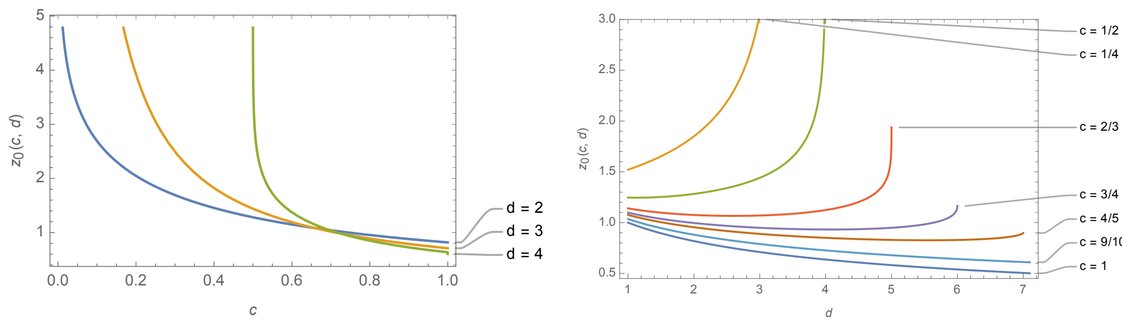

Figure 1. Milne extrapolation distance ("endpoint") for the half space in $\mathbb{R}^d$ for varying c and d.

and

$$N(\nu) = \frac{\nu}{G(\nu)} \left[\lambda(\nu)^2 + \left(\frac{1}{2} c\pi\nu G(\nu) \right)^2 \right]. \tag{27}$$

We have determined various angular moments for monodirectional illumination along cosine $0 < \mu_\ell < 1$,

$$\langle x^n(\mu_\ell) \rangle = \frac{\int_0^\infty x^n dx \int_{-1}^1 I(x,\mu;\mu_\ell) G(\mu) d\mu}{\int_0^\infty dx \int_{-1}^1 I(x,\mu;\mu_\ell) G(\mu) d\mu}, \quad n = 0,\ 1,\ 2, \ldots \tag{28}$$

from which the mean and mean square depths for a photon are

$$\langle x(\mu_\ell) \rangle = \mu_\ell + \frac{c\alpha_1}{2(1-c)^{1/2}}, \quad \langle x^2(\mu_\ell) \rangle = 2\mu_\ell^2 + \frac{c}{(1-c)^{1/2}} \left[\alpha_1 \mu_\ell + \frac{c\alpha_1^2}{2(1-c)^{1/2}} + \alpha_2 \right]. \tag{29}$$

For the Milne problem, the extrapolation distance ("endpoint") for conservative scattering $c = 1$ generalizes to

$$z_0(d) = \frac{1}{\pi} \int_0^\infty \left(\frac{1}{1+t^2} \right) \left(\frac{d}{t^2} + 3 - \frac{1}{1 - {}_2F_1\left(\frac{1}{2}, 1; \frac{d}{2}; -t^2\right)} \right) dt, \tag{30}$$

and, with absorption $(d-3)/(d-2) < c < 1$, we find [4, 5]

$$z_0(c,d) = \frac{\nu_0}{2} \ln \left[\frac{4N(\nu_0) H^2(\nu_0)}{c\,\nu_0} \right] = \frac{\nu_0}{2} \ln \left[\frac{\nu_0 + 1}{\nu_0 - 1} \right] - \frac{1}{\pi} \int_0^1 \frac{\theta(t)}{1 - t^2/\nu_0^2} dt \tag{31}$$

where $\theta(t)$ is determined from Eq.(10).

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